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A SPECIAL CASE OF THE LAPLACE CO-EFFICIENTS $b_s^{(i)}$.

By PROF. ASAPH HALL, Washington, D. C.

(1.) When we consider two planets moving around the sun in the same plane, and take this plane for the plane of reference, the potential, or perturbative function will be, in the notation of the *Mécanique Céleste*,

$$R = \frac{m'r\cos(v'-v)}{r'^2} - \frac{m'}{\sqrt{[r^2 + r'^2 - 2rr'\cos(v'-v)]}}.$$

r, r', v, v' are the polar co-ordinates of the planets referred to the centre of the sun, and m' is the mass of the disturbing planet. If the eccentricities of the orbits are small, the radii vectores will not differ much from the semi-major axes of the orbits. Let these be a and a', and suppose $a < a'$, $\alpha = a / a'$; also let $v' - v = \theta$. Then we have to develop into a periodical series the function,

$$\frac{1}{\sqrt{(1 - 2\alpha\cos\theta + \alpha^2)}}.$$

Omit the subscript s of Laplace, and assume

$$\frac{1}{\sqrt{(1 - 2\alpha\cos\theta + \alpha^2)}} = \tfrac{1}{2}b^{(0)} + b^{(1)}\cos\theta + b^{(2)}\cos 2\theta + b^{(3)}\cos 3\theta + \ldots$$
$$+ b^{(i)}\cos i\theta + \ldots$$

In the *Mécanique Céleste* the values of the $b^{(i)}$ co-efficients are given in series which proceed according to powers of α. For the principal planets of our solar system this method is generally convenient, since the greatest value of α in this system is 0.723, which occurs in the theory of Venus and the Earth. But for values of α greater than this the series converge slowly, and its seems better to determine these co-efficients by a different process. In the second volume of his great work on elliptic integrals, Legendre shows how the $b^{(i)}$ co-efficients may be reduced to these integrals, and their numerical values computed by means of his tables. In this case, however, the integrals can be reduced to forms so convenient that it seems worth while to proceed directly, and without reference to tables.

If we integrate the expression $\cos i\theta \cos j\theta\, d\theta$, between the limits 0 and π, we have,

$$\int_0^\pi \cos i\theta \cos j\theta\, d\theta = \tfrac{1}{2}\int_0^\pi \Big[\cos(i+j)\,\theta + \cos(i-j)\,\theta\Big] d\theta$$

$$= \tfrac{1}{2}\left[\frac{\sin(i+j)\,\theta}{i+j} + \frac{\sin(i-j)\,\theta}{i-j}\right]_0^\pi.$$

Since i and j are whole numbers, the integral becomes zero, except for $i = j$, when its value is $\frac{1}{2}\pi$. If i and j are both zero, the value of the integral is π. Denoting, therefore, the left-hand side of the equation for the series by $f(\theta)$, then multiplying the equation by $\frac{2}{\pi}\cos i\theta\, d\theta$, and integrating from 0 to π, the terms are all zero except

$$b^{(i)} \times \frac{2}{\pi}\int_0^\pi \cos i\theta \cos i\theta\, d\theta.$$

We have, therefore, for the general value of the co-efficient,

$$b^{(i)} = \frac{2}{\pi}\int_0^\pi f(\theta) \cos i\theta\, d\theta. \tag{1}$$

Thus for $i = 0$, and $i = 1$, we have

$$b^{(0)} = \frac{2}{\pi}\int_0^\pi \frac{d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}},$$

$$b^{(1)} = \frac{2}{\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}}.$$

Let us consider three successive co-efficients of the series,

$$b^{(i-1)} = \frac{2}{\pi}\int_0^\pi f(\theta)\cos(i-1)\,\theta\, d\theta,$$

$$b^{(i)} = \frac{2}{\pi}\int_0^\pi f(\theta)\cos i\theta\, d\theta,$$

$$b^{(i+1)} = \frac{2}{\pi}\int_0^\pi f(\theta)\cos(i+1)\,\theta\, d\theta;$$

we can easily find a linear relation between these co-efficients. Integrating by parts the expression

$$\cos i\theta \sqrt{(1 - 2a\cos\theta + a^2)} \,.\, d\theta,$$

we have

$$\int \cos i\theta \sqrt{(1 - 2a\cos\theta + a^2)} \,.\, d\theta$$

$$= \frac{\sin i\theta}{i} \sqrt{(1 - 2a\cos\theta + a^2)} - \frac{a}{i} \,.\, \int \frac{\sin\theta \sin i\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}} \,.$$

The first term is zero at the limits o and π, and since

$$\sin\theta \sin i\theta = \tfrac{1}{2}\left[\cos(i-1)\theta - \cos(i+1)\theta\right],$$

the result is

$$\int_0^\pi \cos i\theta \sqrt{(1 - 2a\cos\theta + a^2)}\, d\theta = -\frac{a\pi}{4i} \,.\, b^{(i-1)} + \frac{a\pi}{4i} \,.\, b^{(i+1)} \,.$$

Treating the same integral in a different manner, we have

$$\int \cos i\theta \sqrt{(1 - 2a\cos\theta + a^2)}\, d\theta$$

$$= (1 + a^2) \int \frac{\cos i\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}} - 2a \int \frac{\cos\theta \cos i\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}}$$

$$= (1 + a^2) \int \frac{\cos i\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}} - a \int \frac{\cos(i+1)\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}}$$

$$- a \int \frac{\cos(i-1)\theta \, d\theta}{\sqrt{(1 - 2a\cos\theta + a^2)}} \,.$$

By comparison of the two values we have

$$-\frac{a}{2i} \,.\, b^{(i-1)} + \frac{a}{2i} \,.\, b^{(i+1)} = (1 + a^2) \,.\, b^{(i)} - a\, b^{(i+1)} - a\, b^{(i-1)},$$

or

$$b^{(i+1)} = \frac{2i}{2i+1} \,.\, \frac{1 + a^2}{a} \,.\, b^{(i)} - \frac{2i-1}{2i+1} \,.\, b^{(i-1)}. \qquad (2)$$

This equation shows that if we know the co-efficients $b^{(i-1)}$ and $b^{(i)}$, we can compute from their values the next co-efficient $b^{(i+1)}$. Hence all the co-efficients may be computed from $b^{(0)}$ and $b^{(1)}$. We can find also from the definite integrals the derivatives of the $b^{(i)}$ co-efficients with respect to a, which are required in the development of the disturbing forces, and these derivatives may be made to de-

pend on the $b^{(i)}$ co-efficients; but these formulæ are well known and appear to be convenient for computation.

(2.) In order that we may reduce the expressions for $b^{(0)}$ and $b^{(1)}$ to forms suitable for numerical computation, we have to transform the integral containing $\sqrt{(1 - 2a\cos\theta + a^2)}$ as a divisor. Since $\cos\theta = 2\cos^2\frac{1}{2}\theta - 1$, we have

$$\sqrt{(1 - 2a\cos\theta + a^2)} = \sqrt{(1 + 2a + a^2 - 4a\cos^2\tfrac{1}{2}\theta)}$$

$$= (1 + a)\sqrt{(1 - \frac{4a}{(1+a)^2}\cos^2\tfrac{1}{2}\theta)}.$$

If, therefore, we put

$$k^2 = \frac{4a}{(1+a)^2}, \qquad a = \frac{1 - \sqrt{(1-k^2)}}{1 + \sqrt{(1-k^2)}},$$

$$x = \cos\tfrac{1}{2}\theta, \qquad d\theta = -\frac{2dx}{\sqrt{(1-x^2)}};$$

the value of $b^{(0)}$ becomes

$$b^{(0)} = \frac{4}{\pi(1+a)}\int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}.$$

This elliptic integral may be transformed according to the method given by Lagrange.* For this purpose put

$$y = x\,.\sqrt{\left(\frac{1-x^2}{1-k^2x^2}\right)}. \tag{3}$$

Then $$\frac{y}{x}(1 - k^2x^2) = \sqrt{[(1-x^2)(1-k^2x^2)]},$$

and $$\frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}} = \frac{x\,dx}{y(1-k^2x^2)}.$$

Equation (3) gives $$y^2 - k^2x^2y^2 = x^2 - x^4,$$

and differentiating, we have first,

$$\frac{x\,dx}{(1-k^2x^2)y} = \frac{dy}{1 + k^2y^2 - 2x^2}.$$

Again, solving (3) for x^2, we find

$$1 + k^2y^2 - 2x^2 = \sqrt{[1 + 2(k^2-2)y^2 + k^4y^4]},$$

so that $$\frac{dy}{1 + k^2y^2 - 2x^2} = \frac{dy}{\sqrt{[1 + 2(k^2-2)y^2 + k^4y^4]}}.$$

Oeuvres de Lagrange, Tome II. p. 253.

Assume the trinomial under the radical to be of the form $(1 + \alpha y^2)(1 + \beta y^2)$, and we shall have

$$\alpha = k^2 - 2 + 2\sqrt{(1 - k^2)},$$
$$\beta = k^2 - 2 - 2\sqrt{(1 - k^2)};$$

and the trinomial is resolved into the real factors,

$$1 + [k^2 - 2 + 2\sqrt{(1 - k^2)}]y^2,$$
$$1 + [k^2 - 2 - 2\sqrt{(1 - k^2)}]y^2.$$

The quantities α and β are of the same sign since their product is k^4. The half sum of these quantities is $k^2 - 2$, which is negative, and therefore α and β are negative.

Let
$$p^2 = 2 - k^2 + 2\sqrt{(1 - k^2)},$$
$$q^2 = 2 - k^2 - 2\sqrt{(1 - k^2)};$$

from which we have

$$p = 1 + \sqrt{(1 - k^2)}, \quad q = 1 - \sqrt{(1 - k^2)}.$$

Thus we have

$$\frac{dx}{\sqrt{[(1 - x^2)(1 - k^2x^2)]}} = \frac{dy}{\sqrt{[(1 - p^2y^2)(1 - q^2y^2)]}}.$$

Now we have
$$p = \frac{2}{1 + a}, \quad q = \frac{2a}{1 + a};$$

and, if in equation (3) we change our unit, or put

$$\frac{1 + a}{2} \cdot y = x \cdot \sqrt{\left(\frac{1 - x^2}{1 - k^2x^2}\right)},$$

our integral becomes

$$\int \frac{dx}{\sqrt{[(1 - x^2)(1 - k^2x^2)]}} = \frac{1 + a}{2} \int \frac{dy}{\sqrt{[(1 - y^2)(1 - a^2y^2)]}}.$$

This last integral is of the same form as the first one, and it is plain that we may continue such transformations as far as we please. Hence if we put as before

$$a = \frac{1 - \sqrt{(1 - k^2)}}{1 + \sqrt{(1 - k^2)}} = \frac{k^2}{[1 + \sqrt{(1 - k^2)}]^2},$$

$$a_1 = \frac{1 - \sqrt{(1 - a^2)}}{1 + \sqrt{(1 - a^2)}} = \frac{a^2}{[1 + \sqrt{(1 - a^2)}]^2},$$

$$a_2 = \frac{1 - \sqrt{(1 - a_1^2)}}{1 + \sqrt{(1 - a_1^2)}} = \frac{a_1^2}{[1 + \sqrt{(1 - a_1^2)}]^2},$$

etc., etc.;

we shall have

$$\int \frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}$$

$$= \frac{1+a}{2} \cdot \frac{1+a_1}{2} \cdot \frac{1+a_2}{2} \cdot \frac{1+a_3}{2} \cdot \cdot \cdot \frac{1+a_n}{2} \cdot \int \frac{dy}{\sqrt{[(1-y^2)(1-a_n^2y^2)]}} \cdot$$

From the values of $a, a_1, a_2, \ldots a_n$, we see that a_n can be diminished at pleasure, and, at last, $a_n^2y^2$ may be neglected. In this case we have

$$\int_0^1 \frac{dy}{\sqrt{(1-y^2)}} = \tfrac{1}{2}\pi.$$

From equation (3) we see that y is zero when $x=0$, and when $x=1$; and that y has a maximum for an intermediate value of x. Hence in the first step of the transformation we must double the integral for the limits, $y=0$ to $y=1$; and the same for each succeeding step; so that in the final value we shall have the factor 2^n. The result will be

$$b^{(0)} = \frac{4}{\pi(1+a)} \cdot \frac{1+a}{2} \cdot \frac{1+a_1}{2} \cdot \frac{1+a_2}{2} \cdot \frac{1+a_3}{2} \cdot \cdot \cdot \frac{1+a_n}{2} \cdot \pi \cdot 2^n,$$

or $$b^{(0)} = 2(1+a_1)(1+a_2)(1+a_3)(1+a_4) \ldots (1+a_n). \qquad (4)$$

The value of $b^{(1)}$ may be found by means of the same substitutions, which give

$$b^{(1)} = \frac{4}{\pi(1+a)} \int_0^1 \frac{(2x^2-1) \cdot dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}} \cdot$$

The preceding reductions give

$$2x^2 - 1 = k^2y^2 - \sqrt{[1 + 2(k^2-2)y^2 + k^4y^4]};$$

and making the same change of the unit as before, we have

$$2x^2 - 1 = \tfrac{1}{2}a(2y^2-1) + \tfrac{1}{2}a - \sqrt{[(1-y^2)(1-a^2y^2)]}.$$

Putting, therefore, for $dx / \sqrt{[(1-x^2)(1-k^2x^2)]}$, its value found before, we have

$$\int_0^1 \frac{(2x^2-1)\,dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}} = \frac{1+a}{2} \cdot \frac{a}{2} \cdot \int_0^1 \frac{(2y^2-1)\,dy}{\sqrt{[(1-y^2)(1-a^2y^2)]}} - \frac{1+a}{2} \int_0^1 dy$$

$$+ \frac{1+a}{2} \cdot \frac{a}{2} \cdot \int_0^1 \frac{dy}{\sqrt{[(1-y^2)(1-a^2y^2)]}} \cdot$$

It is evident that this method of transformation may also be continued, and at last we shall have the forms

$$\int_0^1 \frac{(2x^2-1)\,dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}$$

$$= \frac{1+a}{2}\cdot\frac{1+a_1}{2}\cdot\frac{1+a_2}{2}\cdots\frac{1+a_n}{2}\cdot\frac{a}{2}\cdot\frac{a_1}{2}\cdot\frac{a_2}{2}\cdots\frac{a_n}{2}\cdot\int_0^1\frac{dy}{\sqrt{(1-y^2)}}$$

$$-\frac{1+a}{2}\int_0^1 dy - \frac{1+a}{2}\cdot\frac{1+a_1}{2}\cdot\frac{a}{2}\cdot\int_0^1 dy - \ldots$$

$$+\left\{\begin{array}{l} \dfrac{a\,.\,(1+a)(1+a_1)(1+a_2)\ \ldots\ (1+a_n)}{2^{n+1}} \\ +\dfrac{a\,.\,a_1\,.\,(1+a)(1+a_1)\ \ldots\ (1+a_n)}{2^{n+2}} \\ +\dfrac{a\,.\,a_1\,.\,a_2\,.\,(1+a)(1+a_1)\ \ldots\ (1+a_n)}{2^{n+3}} \\ +\ldots \end{array}\right\}\cdot\int_0^1\frac{dy}{\sqrt{(1-y^2)}}.$$

The first term approaches zero as its limit and may be neglected. The integrals of the second set vanish at the limits zero and unity. We have left only the last terms within the brackets, and since as before the value of the last integral is $\frac{1}{2}\pi\,.\,2^n$, we have finally,

$$b^{(1)} = (1+a_1)(1+a_2)(1+a_3)\ \ldots\ (1+a_n)\left(a+\frac{aa_1}{2}+\frac{aa_1a_2}{2\,.\,2}+\frac{aa_1a_2a_3}{2\,.\,2\,.\,2}+\ldots\right),$$

or

$$b^{(1)} = \tfrac{1}{2}b^{(0)}\left(a+\frac{a\,.\,a_1}{2}+\frac{a\,.\,a_1\,.\,a_2}{2\,.\,2}+\frac{a\,.\,a_1\,.\,a_2\,.\,a_3}{2\,.\,2\,.\,2}+\ \ldots\right). \qquad (5)$$

Equations (4) and (5) can be computed easily, and then the remaining co-efficients may be found from equation (2), when a is not very small.

We may notice here a relation between the moduli of the successive transformations which was pointed out by Lagrange, and which is similar to that employed by Gauss in his celebrated memoir on the secular perturbations of a planet: *Determinatio Attractionis*, etc. 1818.

From the values of a, a_1, a_2, etc. we have

$$a=\frac{q}{p},\quad a_1=\frac{q_1}{p_1},\quad a_2=\frac{q_2}{p_2},\quad \text{etc.};$$

and hence p is greater than q, p_1 greater than q_1, etc.

We have also $p_1 = p + \sqrt{(p^2 - q^2)}$, $q_1 = p - \sqrt{(p^2 - q^2)}$;

$$p_2 = p_1 + \sqrt{(p_1^2 - q_1^2)}, \quad q_2 = p_1 - \sqrt{(p_1^2 - q_1^2)};$$

etc., etc.

If we put

$$p + q = m, \quad p_1 + q_1 = m_1;$$

$$p - q = n, \quad p_1 - q_1 = n_1;$$

etc., etc.;

so that

$$p = \tfrac{1}{2}(m + n), \quad q = \tfrac{1}{2}(m - n);$$

$$p_1 = \tfrac{1}{2}(m_1 + n_1), \quad q_1 = \tfrac{1}{2}(m_1 - n_1);$$

we shall have

$$m_1 = m + n, \quad n_1 = 2\sqrt{(mn)};$$

$$m_2 = m_1 + n_1, \quad n_2 = 2\sqrt{(m_1 n_1)};$$

etc., etc.

In the series of quantities, therefore, $m, m_1, m_2, \ldots; n, n_1, n_2, \ldots$, the corresponding terms are the arithmetical and geometrical means of twice the terms that precede them.

(3.) For the principal planets of our solar system there does not appear to be much need of departing from the methods commonly given for computing the $b^{(i)}$ co-efficients; but in the Saturnian system we find a case where these methods are not convenient. The two satellites of Saturn, Titan and Hyperion, the largest and the smallest of this system, move in the same plane, and from the values of their mean distances from the centre of the planet we have the ratio

$$a = 0.8250863.$$

Hence the usual series for computing the $b^{(i)}$ co-efficients will converge slowly, and the preceding method by definite integrals is much easier. I will compute the values of some of these co-efficients for this value of a. For this purpose put

$$a = \sin\varphi, \quad a_1 = \sin\varphi_1, \quad a_2 = \sin\varphi_2, \quad \text{etc.}$$

The expressions for $a_1, a_2, a_3, \ldots$ give

$$a_1 = \frac{1 - \cos\varphi}{1 + \cos\varphi} = \tan^2\tfrac{1}{2}\varphi,$$

$$a_2 = \frac{1 - \cos\varphi_1}{1 + \cos\varphi_1} = \tan^2\tfrac{1}{2}\varphi_1,$$

$$a_3 = \frac{1 - \cos\varphi_2}{1 + \cos\varphi_2} = \tan^2\tfrac{1}{2}\varphi_2, \quad \text{etc.};$$

and hence

$$1 + a_1 = \sec^2\tfrac{1}{2}\varphi,$$

$$1 + a_2 = \sec^2\tfrac{1}{2}\varphi_1,$$

$$1 + a_3 = \sec^2\tfrac{1}{2}\varphi_2, \quad \text{etc.}$$

The computation of $b^{(0)}$ and $b^{(1)}$ by formulæ (4) and (5) may be arranged as follows:—

$$a = \sin\varphi = 0.8250863,$$
$$\log\sin\varphi = 9.9164993760.$$

Calculation of $b^{(0)}$.

					log		log
φ =	55°	35′	50″.07				
$\frac{1}{2}\varphi$ =	27	47	55.035	tan	9.7219831665	sec	0.0532568941
φ_1 =	16	8	16.349	sin	9.4439663330		
$\frac{1}{2}\varphi_1$ =	8	4	8.174	tan	9.1515781796	sec	0.0043209827
φ_2 =	1	9	5.8228	sin	8.3031563592		
$\frac{1}{2}\varphi_2$ =	0	34	32.9114	tan	8.0021702287	sec	0.0000219319
φ_3 =	0	0	20.8337	sin	6.0043404574		
$\frac{1}{2}\varphi_3$ =	0	0	10.4168	tan	5.7033093	sec	0.0000000005
							0.0575998092
						2 log sec =	0.1151996184
						log 2 =	0.3010299957
						$\log b^{(0)}$ =	0.4162296141
						$b^{(0)}$ =	2.607531805

Calculation of $b^{(1)}$.

$\log a$ =	9.9164993760	a =	0.8250863000
$\log a_1$ =	9.4439663330		
$\log \frac{1}{2}$ =	9.6989700043	2nd term =	0.1146662773
2nd term =	9.0594357133		
$\log a_2$ =	8.3031563592	3rd term =	0.0011522908
$\log \frac{1}{2}$ =	9.6989700043		
3rd term =	7.0615620768	4th term =	0.0000000582
$\log a_3$ =	6.0043404574		0.9409049263
$\log \frac{1}{2}$ =	9.6989700043	log =	9.9735457423
4th term =	2.7648725385	$\log \frac{1}{2}$ =	9.6989700043
$\log a_4$ =	1.4066186	$\log b^{(0)}$ =	0.4162296141
$\log \frac{1}{2}$ =	9.6989700	$\log b^{(1)}$ =	0.0887453607
5th term =	3.8704611 − 20	$b^{(1)}$ =	1.226719759

Having found $b^{(0)}$ and $b^{(1)}$, we can compute the remaining co-efficients by formula (2). Thus

$$b^{(2)} = \tfrac{2}{3} \cdot \frac{1 + a^2}{a} \cdot b^{(1)} - \tfrac{1}{3} \cdot b^{(0)},$$

$$b^{(3)} = \tfrac{4}{5} \cdot \frac{1 + a^2}{a} \cdot b^{(2)} - \tfrac{3}{5} \cdot b^{(1)},$$

$$b^{(4)} = \tfrac{6}{7} \cdot \frac{1 + a^2}{a} \cdot b^{(3)} - \tfrac{5}{7} \cdot b^{(2)},$$

etc., etc.

The preceding calculations would be much shorter if made with seven-figure logarithms, and the accuracy would be sufficient for practical purposes; but the work has been done with ten-figure logarithms in order to show better the rapidity of the convergence. The value of $b^{(0)}$ computed by the common series, including twenty six terms and a^{50}, is

$$b^{(0)} = 2.60752853.$$

The following table contains the values of the first six co-efficients computed by means of the definite integrals and formula (2); and also the values of the same co-efficients computed from the formulæ of Leverrier.*

Values of $b_{\frac{1}{2}}^{(i)}$; $a = 0.8250863$.

Co efficient.	Value by preceding method.	Value by series.
$b^{(0)}$	2.6075318	2.6072929
$b^{(1)}$	1.2267198	1.2265599
$b^{(2)}$	0.7967742	0.7965316
$b^{(3)}$	0.5624428	0.5622774
$b^{(4)}$	0.4129396	0.4126906
$b^{(5)}$	0.3102719	0.3100983

The values computed by Leverrier's formulæ include twelve terms, but they are too small, and it would be necessary to continue the series to double this number of terms if we wish to obtain values accurate in the sixth decimal place. But if series are to be used, they should be transformed so as to proceed according to powers of $1 - a^2$, as was pointed out by Euler in 1750.

(4.) The history of these co-efficients is interesting. Since it is necessary

***Annales de l'Observatoire de Paris*, Tome II. Addition première, p. 5.

to consider the reciprocal of the distance between two bodies when their mutual attractions are to be determined, and the best practicable method of finding these attractions is generally the expansion of this reciprocal into a periodical series, mathematicians were immediately confronted with the problem of finding the co-efficients of the sines and cosines of this series. D'Alembert (*Jean le Rond*) saw in 1743 that all the $b_s^{(i)}$ co-efficients could be found as soon as the first two were known, a relation which is expressed in equation (2). Euler and Clairaut made many improvements in the actual numerical determination of these co-efficients. The elegant method of resolving the general expression of the reciprocal, $(1 - 2a\cos\theta + a^2)^{-s}$, into two binomial factors by means of imaginary quantities, which is employed by Laplace, is due to Lagrange. It is probable that the transformation of the elliptic integral which has been used above was suggested to Lagrange by the work of Landen. It has since been employed by several writers, and among others by Professor Woodhouse.*

In looking at the great amount of labor that has been done on the important problem of developing the perturbative function, one cannot fail to notice the variety of the symbols that have been introduced. This variety has become so great as to cause confusion, and in order to be sure of his formulæ it is now almost necessary that every astronomer should have a development of his own. This condition seems to be unfortunate, since it causes a repetition of work that should be done once for all.

U. S. N. Observatory, Nov. 5, 1886.
Communicated by the Superintendent.

ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

By Mr. R. S. Woodward, Washington, D. C.

[CONTINUED FROM VOL. II, PAGE 131.]

18. We proceed now to the determination of the constants V_0 of equation (6) and U_0 of (56) and (57). It has already been stated that these constants are to be determined from the condition of equality in volumes contained by the disturbed and undisturbed surfaces, a condition whose analytical statement is

$$2\pi r_0^2 \int_0^\pi v \sin a \, da = 0.$$

Philosophical Transactions, London. 1804.

Substituting the value of v from (6) in this, there results

$$\int_0^\pi V \sin a\, da - V_0 \int_0^\pi \sin a\, da = 0,$$

whence
$$V_0 = \tfrac{1}{2} \int_0^\pi V \sin a\, da. \tag{58}$$

The easiest way to evaluate the integral in this equation is to substitute the value of V from equation (54). We get then at once

$$\begin{aligned} V_0 &= 4 r_0 h \rho \pi \sin^2 \tfrac{1}{2}\beta \,.\, \tfrac{1}{2} \int_0^\pi \sin a\, da \\ &= 4 r_0 h \rho \pi \sin^2 \tfrac{1}{2}\beta, \end{aligned} \tag{59}$$

since all terms of the series, except the first, vanish in the integration. In a similar manner it may be shown that

$$U_0 = Y_0 = 2 \sin^2 \tfrac{1}{2}\beta. \tag{60}$$

V_0 may also be derived thus: For points within the perimeter of the attracting mass replace V in (58) by V_1 of (20), and for points outside that perimeter replace V in (58) by V_2 of (21). Making these substitutions there results

$$V_0 = 2 r_0 h \rho \left[\int_0^\beta I_1 \sin a\, da + \int_\beta^\pi I_2 \sin a\, da \right].^* \tag{61}$$

Substituting the value of I_1 from (22),

$$\int_0^\beta I_1 \sin a\, da = 4 \int_0^{\frac{1}{2}\pi} \frac{d\gamma_1}{\sin^2 \gamma_1} \int_0^\beta \left(\frac{\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}a \sin^2 \gamma_1}{1 - \sin^2 \frac{1}{2}a \sin^2 \gamma_1} \right)^{\frac{1}{2}} \sin^2 \gamma_1 d (\sin^2 \tfrac{1}{2}a).$$

*A more direct process than that followed in the text is indicated thus:—

$$\int_0^\beta I_1 \sin a\, da + \int_\beta^\pi I_2 \sin a\, da = -\Big[I_1 \cos a \Big]_0^\beta + \int_0^\beta \cos a \frac{dI_1}{da}\, da, - \Big[I_2 \cos a \Big]_\beta^\pi + \int_\beta^\pi \cos a \frac{dI_2}{da}\, da.$$

Put
$$t^2 = \frac{\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1}{1 - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1}.$$

Then the last integral becomes

$$-4\cos^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \frac{d\gamma_1}{\sin^2 \gamma_1} \int_{t_1}^{t_2} \frac{2t^2 dt}{(t^2-1)^2}$$

$$= 4\cos^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \left(\frac{t_2}{t_2^2 - 1} - \frac{t_1}{t_1^2 - 1} + \tfrac{1}{2}\log_e \frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right) \frac{d\gamma_1}{\sin^2 \gamma_1},$$

in which
$$t_1 = \sin \tfrac{1}{2}\beta,$$
$$t_2 = \frac{\sin \frac{1}{2}\beta \cos \gamma_1}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_1)}}.$$

Substituting these limits in the non-logarithmic part, there results

$$4\sin^2 \tfrac{1}{2}\beta \left(\int_0^{\frac{1}{2}\pi} \frac{d\gamma_1}{\sin \frac{1}{2}\beta \sin^2 \gamma_1} - \frac{1}{\sin \frac{1}{2}\beta} \int_0^{\frac{1}{2}\pi} \sqrt{(1 - \sin^2 \tfrac{1}{2}\beta \sin^2 \gamma_1)} \cdot \frac{\cos \gamma_1}{\sin^2 \gamma_1} d\gamma_1\right)$$

$$+ 4\sin^2 \tfrac{1}{2}\beta \left(\tfrac{1}{2}\cot^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \log_e \frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)} \cdot \frac{d\gamma_1}{\sin^2 \gamma_1}\right).$$

Integrating by parts all terms of this expression except the first, we get

$$4\sin^2 \tfrac{1}{2}\beta \left\{ \begin{aligned} & \frac{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_1)^{\frac{1}{2}}}{\sin \frac{1}{2}\beta \sin \gamma_1} - \frac{\cot \gamma_1}{\sin \frac{1}{2}\beta} \\ & + \text{arc} \sin (\sin \tfrac{1}{2}\beta \sin \gamma_1) \\ & - \tfrac{1}{2}\cot^2 \tfrac{1}{2}\beta \left(\log_e \frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right) \cot \gamma_1 \\ & - \cot^2 \tfrac{1}{2}\beta \ \text{arc} \sin (\sin \tfrac{1}{2}\beta \sin \gamma_1) \end{aligned} \right\}_0^{\frac{1}{2}\pi}.$$

This gives
$$\int_0^{\beta} I_1 \sin \alpha \, d\alpha = 2(\sin \beta - \beta \cos \beta). \qquad (62)$$

The second integral in (61) becomes, by substitution of the value of I_2 from (25),

$$\int_\beta^\pi I_2 \sin\alpha\, d\alpha$$

$$= 4\sin^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \cos^2\gamma_2 d\gamma_2 \int_\beta^\pi \frac{d(\sin^2 \frac{1}{2}\alpha)}{\sqrt{(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\beta \sin^2\gamma_2)}\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2\gamma_2)}}$$

$$= 8\sin^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \left(\cos^2\gamma_2 - \frac{\sin \frac{1}{2}\beta \cos\gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2\gamma_2)}} + \frac{\sin \frac{1}{2}\beta \sin^2\gamma_2 \cos\gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2\gamma_2)}}\right) d\gamma_2$$

$$= 4\sin^2 \tfrac{1}{2}\beta \left\{ \begin{array}{l} \gamma_2 + \frac{1}{2}\sin 2\gamma_2 - 2 \,\text{arc}\sin(\sin \frac{1}{2}\beta \sin\gamma_2) \\ -\dfrac{(1 - \sin^2 \frac{1}{2}\beta \sin^2\gamma_2)^{\frac{1}{2}} \sin\gamma_2}{\sin \frac{1}{2}\beta} \\ +\dfrac{\text{arc}\sin(\sin \frac{1}{2}\beta \sin\gamma_2)}{\sin^2 \frac{1}{2}\beta} \end{array} \right\}_0^{\frac{1}{2}\pi};$$

whence

$$\int_\beta^\pi I_2 \sin\alpha\, d\alpha = 2(\pi \sin^2 \tfrac{1}{2}\beta - \sin\beta + \beta\cos\beta). \qquad (63)$$

The sum of (62) and (63) is $2\pi \sin^2 \frac{1}{2}\beta$, which, substituted in (61) gives the same value for V_0 as (59).

19. By reference now to equations (3), (6), (20) to (23), and (59), we find for the equation of the disturbed surface, when the effect of the rearranged water is neglected,

$$v = 3h \frac{\rho}{\rho_m} \left(\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta\right). \qquad (64)$$

The corresponding expression in polar harmonics is, see equations (56), (57), and (60),

$$v = \tfrac{3}{2}h \frac{\rho}{\rho_m} \sum_{i=0}^{i=\infty} f_i(\cos\alpha)\, F_i(\beta). \qquad (65)$$

Under the assumption that the water covers the whole sphere and is free to adjust itself as stated in §17, the equation to the disturbed surface is

$$v + \Delta v = 3h \frac{\rho}{\rho_m} \left(\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta\right) + \tfrac{3}{2}h \frac{\rho}{\rho_m} \sum_{i=0}^{i=\infty} \frac{f_i(\cos\alpha)\, F_i(\beta)}{(2i+1)\dfrac{\rho_m}{\rho_w} - 3}. \qquad (66)$$

20. The position of any point on the disturbed surface is thus defined by the co-ordinates v and I, or rather v and a, since I is a function of a; v being the elevation or depression of the point relative to the spherical surface, and a the angular distance of the point from the axis of the attracting mass. I is to be computed from (22) or (23), or their equivalents (24) and (25), according as the point is within or without the perimeter of the disturbing mass. The functions which enter the last term of (66) are given by (45) and (47) respectively.

The general nature of the disturbed surface, when the effect of the rearranged water is neglected, is evident from (64). It is symmetrical with respect to the axis of the attracting mass. It lies without or within the spherical surface of reference according as I is greater or less than $\pi \sin^2 \frac{1}{2}\beta$. The values of I for a point of the disturbed surface at the axis of the attracting mass, along its border and opposite the centre of the mass, are given by equations (26), (28), and (30) respectively. If we denote the corresponding values of v by the suffixes 1, 2, 3, we get

$$
\left.
\begin{aligned}
v_1 &= 3h \frac{\rho}{\rho_m} (\sin \tfrac{1}{2}\beta - \sin^2 \tfrac{1}{2}\beta), \\
&\qquad a = 0 \\
v_2 &= 3h \frac{\rho}{\rho_m} \left(\frac{\beta}{\pi} - \sin^2 \tfrac{1}{2}\beta \right), \\
&\qquad a = \beta \\
v_3 &= 3h \frac{\rho}{\rho_m} (2 \sin^2 \tfrac{1}{4}\beta - \sin^2 \tfrac{1}{2}\beta). \\
&\qquad a = \pi
\end{aligned}
\right\} \qquad (67)
$$

The meaning of these equations may be most readily understood by reference to Figure 3. Thus if the circle $EFGH$ represent (in cross-section) the undisturbed sea-level surface of the earth, and a stratum of matter, as an ice cap, $ABDGE$ be added thereto, the new sea-level surface will assume the form indicated by the dotted line. The values of v_1 and v_2 as shown in the diagram are positive, while the value of v_3 is negative. If on the other hand we suppose the space $A'B'D'GFE$ to be filled with matter of less density than the average density of the earth's crust, as is the case in a lake basin, the disturbed or new sea-level surface would fall within some portion PFQ of the undisturbed surface and outside the remaining portion PHQ; i. e. v_1 and v_2 would be negative and v_3 positive.

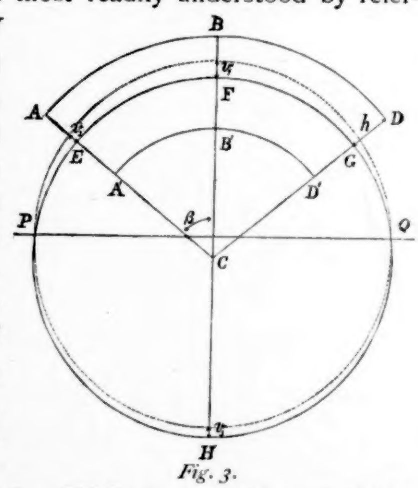

Fig. 3.

21. It is of interest to inquire what angular extent of mass will produce numerical maxima of v_1, v_2, and v_3, supposing the thickness h, and the densities ρ_m and ρ constant. By means of the usual criteria it is readily found that

$$\begin{aligned} v_1 &= \text{a maximum for } \quad \beta = 60^\circ, \\ v_2 &= \text{a maximum for } \sin\beta = 2/\pi, \quad \text{or} \quad \beta = 39^\circ 32', \qquad (68) \\ v_3 &= \text{a maximum for } \quad \beta = 120^\circ. \end{aligned}$$

22. A glance at equation (66) suffices to show that the effect of the free water, if it covers the whole earth, is simply to produce an exaggeration of the type of surface defined by (64) and (65). The series in the third member of (66) expressing this exaggeration is rapidly converging on account of the diminishing factor

$$\frac{\frac{3}{2}}{(2i+1)\frac{\rho_m}{\rho_w} - 3},$$

which is, since ρ_m / ρ_w is about $\frac{11}{2}$,

$$\frac{3}{22i+5}.$$

The essential features of the disturbed surface are, therefore, in any case defined by (64) or its equivalent (65); and in most cases the effect of the rearranged water may be neglected as unimportant or as of no greater magnitude than the uncertainties inherent in the data for actual problems.

23. The equations (67) define the position of the disturbed surface in some of its most characteristic points. To define its position at any other point we must evaluate the elliptic integral I_1 or I_2 which pertains to such point. These integrals have already been expressed, equation (53), in a series of polar harmonics, which, if more convergent, would suffice for computing I_1 or I_2. It is easy, however, to derive more convergent and convenient series than that of (53), and this will be the object of the present section.

First, take I_1 of (24). For brevity put

$$w = \frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\beta} \quad \text{and} \quad b = \sin\tfrac{1}{2}\beta.$$

Then by Maclaurin's series, or by the binominal theorem, we readily find

$$I_1 = 2b\int_0^{\frac{1}{2}\pi}(1 - A\sin^2\gamma_1 - B\sin^4\gamma_1 - C\sin^6\gamma_1 - \ldots)\,d\gamma_1, \qquad (69)$$

in which

$$\begin{aligned} A &= \tfrac{1}{2}w^2(1-b^2), \\ B &= \tfrac{1}{8}w^4(1+2b^2-3b^4), \\ C &= \tfrac{1}{16}w^6(1+b^2+3b^4-5b^6), \\ &\text{etc.} \end{aligned}$$

The even powers of $\sin \gamma_1$ may each be expanded in a series of the form,

$$c + d \cos 2\gamma_1 + e \cos 4\gamma_1 + \ . \ . \ . \ ,$$

in which c, d, e, etc. are constants.

But since
$$\int_0^{\frac{1}{2}\pi} \cos 2n\gamma \, d\gamma = 0,$$

we shall need in these expansions only the values of c. The value of c in the expansion of $\sin^{2n}\gamma$ is

$$c = \frac{2n(2n-1)(2n-2) \ . \ . \ . \ (n+1)}{1 \ . \ 2 \ . \ 3 \ . \ . \ . \ n} \cdot \left(\frac{1}{2}\right)^{2n}.$$

Applying this formula, and making the integration in (69), there results

$$I = b\pi\left(1 - \tfrac{1}{2}A - \tfrac{3}{8}B - \tfrac{5}{16}C - \ . \ . \ . \ \right).$$

Hence if we put*

$$\begin{aligned}
g_1 &= \tfrac{1}{4}(1-b^2), \\
g_2 &= \tfrac{3}{64}(1-b^2)(1 + 3b^2), \\
g_3 &= \tfrac{5}{256}(1-b^2)(1 + 2b^2 + 5b^4), \\
g_4 &= \tfrac{35}{16384}(1-b^2)(5 + 9b^2 + 15b^4 + 35b^6), \\
g_5 &= \tfrac{63}{65536}(1-b^2)(7 + 12b^2 + 18b^4 + 28b^6 + 63b^8), \\
g_6 &= \tfrac{231}{1048576}(1-b^2)(21 + 35b^2 + 50b^4 + 70b^6 + 105b^8 + 231b^{10}), \\
&\text{etc.,}
\end{aligned}$$

$$I_1 = b\pi\left(1 - g_1 w^2 - g_2 w^4 - g_3 w^6 - \ . \ . \ . \ \right). \qquad (70)$$

$$w = \frac{\sin \frac{1}{2}\alpha}{\sin \frac{1}{2}\beta}, \quad b = \sin \tfrac{1}{2}\beta, \quad \alpha \leqq \beta.$$

This series converges rapidly except for values of w near unity. In an important practical application, wherein $\beta = 38^\circ$, (70) gives, using

* The general value of g is

$$g_n = \frac{2n(2n-1)(2n-2) \ . \ . \ . \ (n+1)}{1^2 \ . \ 2^2 \ . \ 3^2 \ . \ . \ . \ n^2 \ . \ 2^{3n}} \left\{ \begin{array}{l} 1 \ . \ 1 \ . \ 3 \ . \ 5 \ . \ . \ . \ (2n-3) \\ + 1 \ . \ 1 \ . \ 3 \ . \ 5 \ . \ . \ . \ (2n-5) \ 1 \ . \ nb^2 \\ + 1 \ . \ 1 \ . \ 3 \ . \ 5 \ . \ . \ . \ (2n-7) \ 1 \ . \ 3 \ \dfrac{n(n-1)}{1 \ . \ 2} b^4 \\ + 1 \ . \ 1 \ . \ 3 \ . \ 5 \ . \ . \ . \ (2n-9) \ 1 \ . \ 3 \ . \ 5 \ \dfrac{n(n-1)(n-2)}{1 \ . \ 2 \ . \ 3} b^6 \\ + \ . \ . \ . \\ - 1 \ . \ 3 \ . \ 5 \ . \ . \ . \ (2n-1) \ b^{2n} \end{array} \right\}.$$

terms up to that in w^{12} inclusive, I_1 too great by about 5 per cent. for the case $w = 1$. But this is the most unfavorable case, and one, moreover, for which the exact value of I_1 is known from equation (28).

By a process entirely similar to that followed above, the expansion of (25) gives,* writing for brevity,

$$\nu = \operatorname{cosec} \tfrac{1}{2}a,$$

$$I_2 = \tfrac{1}{2}b^2\pi r \left\{ \begin{array}{l} 1 + \tfrac{1}{8}b^2 \; (1 + \nu^2) \\ + \tfrac{1}{64}b^4 \; (3 + 2\nu^2 + 3\nu^4) \\ + \tfrac{5}{1024}b^6 \; (5 + 3\nu^2 + 3\nu^4 + 5\nu^6) \\ + \tfrac{7}{16384}b^8 \; (35 + 20\nu^2 + 18\nu^4 + 20\nu^6 + 35\nu^8) \\ + \tfrac{21}{131072}b^{10} (63 + 35\nu^2 + 30\nu^4 + 30\nu^6 + 35\nu^8 + 63\nu^{10}) \\ + \; . \; . \; . \end{array} \right\}.$$

If in this expression we put

$$\begin{aligned} k_1 &= \tfrac{1}{2} + \tfrac{1}{16}b^2 + \tfrac{3}{128}b^4 + \tfrac{25}{2048}b^6 + \tfrac{245}{32768}b^8 + \tfrac{1323}{262144}b^{10} + \ldots, \\ k_2 &= \tfrac{1}{16} + \tfrac{1}{64}b^2 + \tfrac{15}{2048}b^4 + \tfrac{35}{4096}b^6 + \tfrac{735}{262144}b^8 + \ldots, \\ k_3 &= \tfrac{3}{128} + \tfrac{15}{2048}b^2 + \tfrac{63}{16384}b^4 + \tfrac{315}{131072}b^6 + \ldots, \\ k_4 &= \tfrac{25}{2048} + \tfrac{35}{4096}b^2 + \tfrac{315}{131072}b^4 + \ldots, \\ k_5 &= \tfrac{245}{32768} + \tfrac{735}{262144}b^2 + \ldots, \\ k_6 &= \tfrac{1323}{262144} + \ldots, \end{aligned}$$

etc.,

we find
$$I_2 = b\pi\,(k_1 w^{-1} + k_2 w^{-3} + k_3 w^{-5} + \; . \; . \; . \;). \tag{71}$$

$$w = \frac{\sin \frac{1}{2}a}{\sin \frac{1}{2}\beta}, \quad b = \sin \tfrac{1}{2}\beta, \quad a \mathrel{\overline{\gtrless}} \beta.$$

This series converges somewhat more rapidly than (70). For the case in which $\beta = 38^\circ$, and for the extreme value $w = 1$, using terms to that in w^{-11} inclusive, (71) gives I_2 too small by about 3 per cent.

*The general term of I_2 is

$$\frac{1}{4} \cdot \frac{2n\,(2n-1)\,(2n-2) \; . \; . \; . \; (n+2)}{1^2 \,.\, 2^2 \,.\, 3^2 \; . \; . \; . \; n^2 \,.\, 2^{3n}} \left\{ \begin{array}{l} 1 \cdot 3 \cdot 5 \ldots (2n-1) \\ + 1 \cdot 3 \cdot 5 \ldots (2n-3)\, 1 \,.\, n\nu^2 \\ + 1 \cdot 3 \cdot 5 \ldots (2n-5)\, 1 \,.\, 3 \, \dfrac{n\,(n-1)}{1 \,.\, 2}\, \nu^4 \\ + 1 \cdot 3 \cdot 5 \ldots (2n-7)\, 1 \cdot 3 \cdot 5 \, \dfrac{n\,(n-1)\,(n-2)}{1 \,.\, 2 \,.\, 3}\, \nu^6 \\ + \; \ldots \\ + 1 \cdot 3 \cdot 5 \ldots (2n-1)\, \nu^{2n} \end{array} \right\} b^{2n}.$$

For points near the border of the disturbing mass, I_2 may be expressed by a more rapidly converging series than (71). Thus from equation (23)

$$I_2 = \int_0^\beta \sqrt{\left(1 - \frac{\cos\beta - \cos\alpha}{\cos p - \cos\alpha}\right)} \cdot dp.$$

Let $\qquad \cos\beta - \cos\alpha = 2 \sin\tfrac{1}{2}(\alpha + \beta) \sin\tfrac{1}{2}(\alpha - \beta) = 2a.$

Then

$$I_2 = \int_0^\beta \left(1 - \frac{a}{\cos p - \cos\alpha} - \frac{\tfrac{1}{2}a^2}{(\cos p - \cos\alpha)^2} - \frac{\tfrac{1}{3}a^3}{(\cos p - \cos\alpha)^3} - \ldots\right) dp.$$

Now if
$$X = \int_0^\beta \frac{dp}{\cos p - \cos\alpha}$$

$$= \frac{1}{\sin\alpha} \log_e \frac{\sin\tfrac{1}{2}(\alpha + \beta)}{\sin\tfrac{1}{2}(\alpha - \beta)};$$

$$\frac{dX}{d\alpha} = -\sin\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2},$$

$$\frac{d^2X}{d\alpha^2} = -\cos\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2} + 2\sin^2\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^3},$$

etc.;

whence
$$\int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2} = -\frac{1}{\sin\alpha} \frac{dX}{d\alpha},$$

$$\int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^3} = \frac{1}{2\sin^2\alpha}\left(\frac{d^2X}{d\alpha^2} - \cot\alpha \frac{dX}{d\alpha}\right),$$

etc.

The integrals in the third and higher terms of the above series are thus seen to depend on the integral in the second term. Making the requisite differentiations we find to terms of the third order inclusive,

$$I_2 = \beta - \left(\frac{a}{\sin\alpha} + \frac{a^2\cos\alpha}{2\sin^3\alpha} + \frac{a^3(3 - 2\sin^2\alpha)}{4\sin^5\alpha} + \ldots\right) \log_e \frac{\sin\tfrac{1}{2}(\alpha + \beta)}{\sin\tfrac{1}{2}(\alpha - \beta)}$$
$$- \left(\frac{5a\sin\beta}{16\sin^2\alpha} + \frac{3a^4\cos\alpha\sin\beta}{8\sin^4\alpha} + \ldots\right), \qquad (72)$$

$$a = \sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha - \beta).$$

24. Having derived the requisite formulæ for computing the position of any point of the disturbed surface, it remains to determine the slope of this surface relative to the undisturbed surface.

Differentiating equation (64) with respect to a, and dividing the result by the radius of the undisturbed surface r_0, we get

$$\frac{dv}{r_0\, da} = \frac{3h\rho}{r_0 \pi \rho_m} \cdot \frac{dI}{da}. \tag{73}$$

This expresses the slope or inclination of the disturbed to the undisturbed surface in a meridian plane through the centre of the disturbing mass; it also expresses the deflection of the plumb-line in the same plane.

In order to apply (73), it is essential to have the general value of

$$\frac{dI}{da}.$$

Since

$$w^n = \left(\frac{\sin \frac{1}{2}a}{\sin \frac{1}{2}\beta}\right)^n,$$

$$\frac{dw^n}{da} = \tfrac{1}{2}n\, w^n \cot \tfrac{1}{2}a;$$

and hence (70) gives

$$\frac{dI}{da} = -\pi \cos \tfrac{1}{2}a \left\{\begin{array}{l} 1g_1 w^1 \\ +\, 2g_2 w^3 \\ +\, 3g_3 w^5 \\ +\, 4g_4 w^7 \\ +\, \ldots \end{array}\right\}. \tag{74}$$

Similarly (71) gives

$$\frac{dI_2}{da} = -\pi \cos \tfrac{1}{2}a \left\{\begin{array}{l} \frac{1}{2}k_1 w^{-2} \\ +\, \frac{3}{2}k_2 w^{-4} \\ +\, \frac{5}{2}k_3 w^{-6} \\ +\, \frac{7}{2}k_4 w^{-8} \\ +\, \ldots \end{array}\right\}. \tag{75}$$

Equations (74) and (75) will suffice for the computation of dI/da except for points near to or at the border of the attracting mass. As a approaches equality to β the above series become less and less convergent, and finally divergent

when $a = \beta$, or $w = 1$. This may be most readily seen by differentiating (22) or (23) with respect to a and then making $a = \beta$. Thus we find

$$\frac{dI}{da} = -\tfrac{1}{2}\sin\beta \int_0^\beta \frac{dp}{\cos\beta - \cos p} = -\frac{1}{2}\left[\log_e \frac{\sin\frac{1}{2}(\beta + p)}{\sin\frac{1}{2}(\beta - p)}\right]_0^\beta = -\infty.$$

Likewise the integrals (24) and (25) become after differentiating them with respect to a and then making $a = \beta$,

$$\frac{dI_1}{da} = -\int_0^{\frac{1}{2}\pi} \frac{\sec^2\gamma_1 \tan\gamma_1 \, d\gamma_1}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_1)^3}}, \tag{A}$$

$$\frac{dI_2}{da} = -\int_0^{\frac{1}{2}\pi} \frac{\sec^2\gamma_2 \, d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)}} \tag{B}$$

$$= -\int_0^{\frac{1}{2}\pi} \frac{\sec^2\frac{1}{2}\beta \sec^2\gamma_2 \, d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)^3}} - \int_0^{\frac{1}{2}\pi} \frac{\sec^2\gamma_2 \tan^2\gamma_2 \, d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)^3}}$$

$$= -1 + \frac{dI_1}{da}.$$

This shows the equality of (A) and (B), since (B) is plainly infinite, its value being

$$-\left[\log_e \frac{\tan\gamma_2 + \sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)}}{\sec^2\frac{1}{2}\beta}\right]_0^{\frac{1}{2}\pi}.$$

25. This failure of equations (74) and (75) for points at the border of the attracting mass arises from the fact that the expressions (20) and (21), though very approximate for the magnitude of the potential V, are not sufficiently general to give an accurate value of dV/da, or the attraction in the direction of the arc a, for those points. To determine the slope of the disturbed surface at the immediate border of the disturbing mass a special investigation is requisite.

Since by equations (3) and (6) the slope is expressed by

$$\frac{dv}{r_0 da} = \frac{3}{4} \cdot \frac{1}{r_0 \pi \rho_m} \cdot \frac{dV}{r_0 da}, \tag{76}$$

we may derive an expression for the attraction $dV/r_0 da$ directly. The exact ex-

pression for the horizontal attraction towards the axis of the mass, of any element-mass is, using the same notation as in § 7,

$$-\rho\frac{4r^3\,dr\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\,d\theta\cos\lambda\,d\lambda}{\sqrt{[(r-r')^2+4rr'\sin^2\tfrac{1}{2}\theta]^3}};$$

and the integral of this is $dV/r_0\,d\alpha$.

Now, as heretofore, let $\quad r = r_0 + u,$

$$r' = r_0 + v.$$

In addition put $\quad \eta = r - r',$

so that $\quad d\eta = dr;$

$$\eta = -v, \quad \text{for} \quad r = r_0,$$

$$= h - v, \quad \text{for} \quad r = r_0 + h.$$

Also let $\quad \xi = 2r_0\sin\tfrac{1}{2}\theta,$

whence $\quad d\xi = r_0\cos\tfrac{1}{2}\theta\,d\theta,$

$$\cos\tfrac{1}{2}\theta = \sqrt{\left[1-\left(\frac{\xi}{2r_0}\right)^2\right]}.$$

Making the substitutions and neglecting terms of the order

$$\frac{u}{r_0},\quad \frac{v}{r_0},\quad \text{and}\quad \left(\frac{\xi}{2r_0}\right)^2,$$

the above expression becomes

$$\rho\frac{\xi^2\,d\xi\,d\eta\cos\lambda\,d\lambda}{\sqrt{(\xi^2+\eta^2)^3}}.$$

Integrating with respect to η and substituting the limits given above, there results

$$\left(\frac{(h-v)\,d\xi}{\sqrt{[\xi^2+(h-v)^2]}}+\frac{v\,d\xi}{\sqrt{(\xi^2+v^2)}}\right)\cos\lambda\,d\lambda.$$

If now we suppose the attracted point on the border of the attracting mass, the limits of ξ will be 0, and, with sufficient approximation, $2r_0\sin\beta\cos\lambda = c\cos\lambda$, say. Integrating with respect to ξ and substituting these limits, we get

$$\rho(h-v)\cos\lambda\,d\lambda\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{(h-v)^2}\right)}+\frac{c\cos\lambda}{h-v}\right]$$

$$+\rho v\cos\lambda\,d\lambda\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{v^2}\right)}+\frac{c\cos\lambda}{v}\right].$$

It remains to integrate these last expressions with respect to λ between the limits

o and $\frac{1}{2}\pi$. An application of the formula for integration by parts will readily transform them to elliptics, but since their element functions decrease very rapidly from the lower to the upper limit, the following process* will suffice. Consider the integral

$$\int_0^{\lambda} \cos\lambda \, d\lambda \log_e \left[\sqrt{\left(1 + \frac{c^2\cos^2\lambda}{v^2}\right)} + \frac{c\cos\lambda}{v}\right],$$

in which λ is such that $\left(\frac{v}{c\cos\lambda}\right)^2$ may be neglected in comparison with unity. In the cases we have to consider $\left(\frac{v}{c\cos\lambda}\right)^2$ will not exceed $\frac{1}{100}$ if $\cos\lambda = \frac{1}{100}$, or $\lambda = 89^\circ\ 25'$, about. Then since

$$\log_e\left[\sqrt{\left(1 + \frac{c^2\cos^2\lambda}{v^2}\right)} + \frac{c\cos\lambda}{v}\right]$$

$$= \log_e\left[\frac{2c\cos\lambda}{v}\left(1 + \frac{1}{4}\left(\frac{v}{c\cos\lambda}\right)^2 + \ldots\right)\right],$$

the above integral becomes

$$\int_0^{\lambda} \cos\lambda \, d\lambda \log_e \frac{2c\cos\lambda}{v} = \log_e \frac{2c}{v}\int_0^{\lambda}\cos\lambda\, d\lambda + \int_0^{\lambda}\cos\lambda\, d\lambda \log_e \cos\lambda$$

$$= \sin\lambda\left(\log_e \frac{2c}{v} - 1\right) + \log_e(1 + \sin\lambda)$$

$$+ (\sin\lambda - 1)\log\cos\lambda.$$

But since $\sin\lambda$ is very nearly unity, the last expression reduces to

$$\log_e\left(\frac{4c}{v}\right) - 1.$$

The error of this integral arising from the use of λ instead of $\frac{1}{2}\pi$ as the upper limit, is less than

$$(\tfrac{1}{2}\pi - \lambda)\cos\lambda \log_e\left[\sqrt{\left(1 + \frac{c^2\cos^2\lambda}{v^2}\right)} + \frac{c\cos\lambda}{v}\right],$$

which, if $\cos\lambda = \frac{1}{100}$ and $c\cos\lambda / v = 10$, amounts to about $\frac{1}{3000}$.

*Given in a somewhat different form by Helmert in *Theorieen der Höheren Geodäsie*, Vol. II. p. 322.

For the entire attraction, therefore, of the mass for a point on its border we get

$$\frac{dV}{r_0 d\alpha} = -2\rho\left[(h-v)\left(\log_e\frac{4c}{h-v}-1\right)+v\left(\log_e\frac{4c}{v}-1\right)\right]$$

$$= -2\rho\left[h\left(\log_e\frac{4c}{h-v}-1\right)+v\log_e\frac{h-v}{v}\right].$$

Finally, restoring in the last expression the value of c, viz. $c = 2r_0 \sin\beta$, (76) becomes

$$\frac{dv}{r_0 d\alpha} = -\frac{3}{2}\frac{h\rho}{r_0\pi\rho_m}\left(\log_e\frac{8r_0\sin\beta}{h-v}+\frac{v}{h}\log_e\frac{h-v}{v}-1\right). \qquad (77)$$

26. Thus far the disturbed surface has been referred to a spherical surface concentric with the earth's centre of gravity before the disturbance arose. In determining the effects of the ice-mass in glacial times, this is the proper surface of reference, since we wish to know the distortion of the sea-level in those times relative to the sea-level in preceding and following epochs. If, however, it is desired to consider the joint effect in disturbing the sea-level of existing masses, like the continents, on the hypothesis that such masses rest on the surface of a centrobaric sphere, a better surface of reference will obviously be the disturbed or existing centre of gravity of the earth. The use of the latter centre will require a slight modification of the preceding formulæ defining the disturbed sea-surface.

To determine the radial displacement of the earth's centre of gravity due to the addition of such a superficial mass as we have considered, it is only necessary to equate the statical moment of that mass to the statical moment of the earth's mass, the moment plane being perpendicular to the axis of the disturbing mass at the undisturbed centre of gravity of the earth. The moment of an elementary ring of angular radius β', measured from the axis of the disturbing mass, is to our order of approximation,

$$2r_0^3 h\rho\pi \sin\beta' \cos\beta' \, d\beta'.$$

Hence if σ denote the displacement sought and M the earth's mass,

$$M\sigma = r_0^3 h\rho\pi \int_0^\beta 2\sin\beta'\cos\beta'\,d\beta'$$

$$= r_0^3 h\rho\pi \sin^2\beta.$$

Therefore, by substitution of the value of M given in equation (3), we find

$$\sigma = \frac{3}{4}h\frac{\rho}{\rho_m}\sin^2\beta. \qquad (78)$$

Now the elevation of any point of the disturbed surface relative to the sphere in the new position will be less than its elevation relative to the sphere in the former position by an amount whose value to the proper degree of approximation is

$$\sigma\cos\alpha,$$

α being as heretofore the angular distance of the point from the axis of the disturbing mass. That is, if v' denote what v becomes by the change in position of the sphere of reference,

$$v' = v - \sigma \cos \alpha.$$

Hence, by virtue of (64) and (78), we find for the equation of the disturbed surface when the sphere of reference is concentric with the disturbed centre of gravity of the earth,

$$v' = 3h \frac{\rho}{\rho_m} \left[\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta \left(1 + \cos \alpha \cos^2 \tfrac{1}{2}\beta \right) \right]; \tag{79}$$

and the slope of the disturbed surface with respect to the surface of reference is

$$\frac{dv'}{r_0 d\alpha} = 3h \frac{\rho}{r_0 \pi \rho_m} \left(\frac{dI}{d\alpha} + \tfrac{1}{4}\pi \sin \alpha \sin^2 \beta \right). \tag{80}$$

27. Thus far the thickness of the disturbing mass has been considered uniform. To determine the effect of a mass symmetrical about an axis, but of variable thickness, we may proceed thus: firstly, suppose the effect of the re-arranged water neglected. Then the differential of equation (64) with respect to β gives

$$\frac{dv}{d\beta} d\beta = \frac{3h\rho}{\pi \rho_m} \cdot \frac{d(I - \pi \sin^2 \frac{1}{2}\beta)}{d\beta} d\beta. \tag{81}$$

This expresses the elevation of the disturbed surface due to an annulus of angular radius β, of angular width $d\beta$, and of height h, the density ρ being uniform. If in this equation we make h a function of β, or write $h = \varphi(\beta)$, and integrate between the proper limits, the result will be the elevation of the disturbed surface due to a mass whose thickness conforms to the law expressed by $\varphi(\beta)$. Calling, for the sake of distinction, the new value of the elevation of the disturbed surface v'', and the proper limits of β, β_1 and β_2, the result of this integration is

$$v'' = \frac{3\rho}{\pi \rho_m} \int_{\beta_1}^{\beta_2} \frac{d(I - \pi \sin^2 \frac{1}{2}\beta)}{d\beta} \varphi(\beta) \, d\beta. \tag{82}$$

Secondly, if the effect of the re-arranged water be taken into account, we must add to v'' of (82) the following increment obtained from the third member of (66):—

$$\Delta v'' = \frac{9\rho}{2\rho_m} \sum_{i=1}^{i=\infty} \left\{ \frac{f_i(\cos \alpha) \displaystyle\int_{\beta_1}^{\beta_2} \frac{dF_i(\beta)}{d\beta} \varphi(\beta) \, d\beta}{(2i+1)\dfrac{\rho_m}{\rho_w} - 3} \right\}. \tag{83}$$

The equations (82) and (83) assign the effect of any homogeneous mass symmetrical with respect to an axis, subject to the restriction that the maximum thickness of the mass may be neglected in comparison with the radius of the earth.

The integral in (82) depends on, and will in general be, no less complex than I, which is defined by equations (22) to (25). In the application of (83) it is to be observed that by (46)

$$\frac{dF_i(\beta)}{d\beta} = f_i(\cos\beta) \cdot \sin\beta.$$

SOLUTIONS OF EXERCISES.

94

O is the centre of the circumscribed circle of ABC, and D, E, F the middle points of its sides. Show that

$$OD^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

SOLUTION.

Let H be the orthocentre, and A', B', C' the feet of the perpendiculars. Since $OD = R\cos A$, etc.,

$$OD^2 + OE^2 + OF^2 = R^2(\cos^2 A + \cos^2 B + \cos^2 C)$$
$$= R^2(1 - 2\cos A\cos B\cos C).$$

But $\quad R = 2R', \quad$ and $\quad r' = A'H\cos A$

$$= BH\cos A\cos C$$
$$= 2R\cos A\cos B\cos C;$$

$$\therefore\quad OD^2 + OE^2 + OF^2 = R^2\left(1 - \frac{r'}{R}\right) = 2R'(2R' - r').$$

[*Marcus Baker.*]

97

IN the triangle ABC two lines drawn from C trisect the side AB. Given c, C, and the angle φ between the trisecants; to solve the triangle.

[*Marcus Baker.*]

SOLUTION.

Let M, N be the points of trisection. Then the anharmonic ratios of the row $AMNB$ and the pencil C–$AMNB$ being equal, we have

$$\frac{\sin ACM \,.\, \sin BCN}{\sin ACB \,.\, \sin MCN} = \frac{AM \,.\, BN}{AB \,.\, MN},$$

or, with the notations given,

$$\sin x \sin y = \tfrac{1}{3} \sin C \sin \varphi,$$

where $$x = ACM, \quad y = BCN.$$

We have also $$x + y = C - \varphi.$$

But since $$\cos(x - y) = \cos(x + y) + 2 \sin x \sin y,$$

these relations give

$$\cos(x - y) = \tfrac{4}{3} \cos(C - \varphi) - \tfrac{1}{3} \cos(C + \varphi);$$

whence $x - y$ is to be found, and thence x and y. A and B are then found from the relations

$$\tan \tfrac{1}{2}(A - N) = \frac{\tan \frac{1}{2}(x - \varphi)}{\tan^2 \frac{1}{2}(x + \varphi)}, \quad A + N = \pi - x - \varphi;$$

$$\tan \tfrac{1}{2}(B - M) = \frac{\tan \frac{1}{2}(y - \varphi)}{\tan^2 \frac{1}{2}(y + \varphi)}, \quad B + M = \pi - y - \varphi.$$

[*J. E. Hendricks.*]

98

THE eccentric anomalies of three points on an ellipse are p_1, p_2, p_3. Show that the area of their triangle is

$$\Delta = 2ab \sin \frac{p_2 - p_3}{2} \sin \frac{p_3 - p_1}{2} \sin \frac{p_1 - p_2}{2};$$

the centre of its circumscribing circle

$$x = + \frac{c^2}{a} \cos \frac{p_2 + p_3}{2} \cos \frac{p_3 + p_1}{2} \cos \frac{p_1 + p_2}{2},$$

$$y = - \frac{c^2}{b} \sin \frac{p_2 + p_3}{2} \sin \frac{p_3 + p_1}{2} \sin \frac{p_1 + p_2}{2};$$

and hence show that the centre of curvature of the ellipse at (x, y) is

$$X = + \frac{c^2 x^3}{a^4}, \quad Y = - \frac{c^2 y^3}{b^4}.$$

[*W. M. Thornton.*]

SOLUTION I.

The co-ordinates of the three points are

$$x_1 = a\cos p_1,\quad y_1 = b\sin p_1;$$
$$x_2 = a\cos p_2,\quad y_2 = b\sin p_2;$$
$$x_3 = a\cos p_3,\quad y_3 = b\sin p_3.$$

Then* $\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{vmatrix}$

$$= \tfrac{1}{2}ab\begin{vmatrix} \cos p_2 - \cos p_1 & \sin p_2 - \sin p_1 \\ \cos p_3 - \cos p_2 & \sin p_3 - \sin p_2 \end{vmatrix}$$

$$= -2ab\begin{vmatrix} \sin\tfrac{1}{2}(p_1+p_2)\sin\tfrac{1}{2}(p_1-p_2) & \cos\tfrac{1}{2}(p_1+p_2)\sin\tfrac{1}{2}(p_1-p_2) \\ \sin\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_2-p_3) & \cos\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_2-p_3) \end{vmatrix}$$

$$= 2ab\sin\tfrac{1}{2}(p_1-p_2)\sin\tfrac{1}{2}(p_2-p_3)\sin\tfrac{1}{2}(p_3-p_1).$$

The co-ordinates of the centre† are

$$x = \tfrac{1}{4}\begin{vmatrix} 1 & y_1 & r_1^2 \\ 1 & y_2 & r_2^2 \\ 1 & y_3 & r_3^2 \end{vmatrix} \div \Delta = \tfrac{1}{4}\alpha / \Delta,$$

$$y = -\tfrac{1}{4}\begin{vmatrix} 1 & x_1 & r_1^2 \\ 1 & x_2 & r_2^2 \\ 1 & x_3 & r_3^2 \end{vmatrix} \div \Delta = \tfrac{1}{4}\beta / \Delta.$$

Substituting the values of the co-ordinates of the three points,

$$\alpha = b\begin{vmatrix} 1 & \sin p_1 & a^2 - (a^2-b^2)\sin^2 p_1 \\ 1 & \sin p_2 & a^2 - (a^2-b^2)\sin^2 p_2 \\ 1 & \sin p_3 & a^2 - (a^2-b^2)\sin^2 p_3 \end{vmatrix}$$

$$= -b(a^2-b^2)\begin{vmatrix} 1 & \sin p_1 & \sin^2 p_1 \\ 1 & \sin p_2 & \sin^2 p_2 \\ 1 & \sin p_3 & \sin^2 p_3 \end{vmatrix}$$

$$= b(a^2-b^2)(\sin p_1 - \sin p_2)(\sin p_2 - \sin p_3)(\sin p_3 - \sin p_1)$$

$$= 8b(a^2-b^2)\cos\tfrac{1}{2}(p_1+p_2)\cos\tfrac{1}{2}(p_2+p_3)\cos\tfrac{1}{2}(p_3+p_1)\sin\tfrac{1}{2}(p_1-p_2)$$
$$\sin\tfrac{1}{2}(p_2-p_3)\sin\tfrac{1}{2}(p_3-p_1).$$

*Salmon's *Conic Sections*, § 36.
†Salmon's *Conic Sections*, §§ 80, *94.

Similarly

$$\beta = a \begin{vmatrix} 1 & \cos p_1 & b^2 + (a^2 - b^2)\cos^2 p_1 \\ 1 & \cos p_2 & b^2 + (a^2 - b^2)\cos^2 p_2 \\ 1 & \cos p_3 & b^2 + (a^2 - b^2)\cos^2 p_3 \end{vmatrix}$$

$$= -8a(a^2 - b^2)\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_3 + p_1)\sin\tfrac{1}{2}(p_1 - p_2)$$
$$\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_3 - p_1).$$

Hence, if $c^2 = a^2 - b^2$,

$$x = +\frac{c^2}{a}\cos\tfrac{1}{2}(p_1 + p_2)\cos\tfrac{1}{2}(p_2 + p_3)\cos\tfrac{1}{2}(p_3 + p_1).$$

$$y = -\frac{c^2}{b}\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_3 + p_1).$$

If the three points coincide, these become

$$X = +\frac{c^2}{a}\cos^3 p = +\frac{c^2x^3}{a^4},$$

$$Y = -\frac{c^2}{b}\sin^3 p = -\frac{c^2y^3}{b^4}.$$

[*Ormond Stone.*]

SOLUTION II.

$$2\Delta = (y_1 - y_2)(x_3 - x_2) - (x_1 - x_2)(y_3 - y_2)$$
$$= ab[(\sin p_1 - \sin p_2)(\cos p_3 - \cos p_2) - (\sin p_3 - \sin p_2)(\cos p_1 - \cos p_2)]$$
$$= 4ab[-\cos\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_1 - p_2)\sin\tfrac{1}{2}(p_3 + p_2)\sin\tfrac{1}{2}(p_3 - p_2)$$
$$+ \cos\tfrac{1}{2}(p_3 + p_2)\sin\tfrac{1}{2}(p_3 - p_2)\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_1 - p_2)]$$
$$= 4ab\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_1 - p_2)[\sin\tfrac{1}{2}(p_2 + p_3)\cos\tfrac{1}{2}(p_1 + p_2)$$
$$- \cos\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_1 + p_2)]$$
$$= 4ab\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_1 - p_2)\sin\tfrac{1}{2}(p_3 - p_1).$$

The equations of the perpendiculars to the chords at their middle points are

$$y - \tfrac{1}{2}(y_1 + y_2) = -\frac{x_2 - x_1}{y_2 - y_1}[x - \tfrac{1}{2}(x_1 + x_2)]$$

$$y - \tfrac{1}{2}(y_2 + y_3) = -\frac{x_3 - x_2}{y_3 - y_2}[x - \tfrac{1}{2}(x_2 + x_3)].$$

Eliminating y, we have

$$\tfrac{1}{2}(y_3 - y_1) = x\left(\frac{x_3 - x_2}{y_3 - y_2} - \frac{x_2 - x_1}{y_2 - y_1}\right) + \frac{(x_2 - x_1)(x_2 + x_1)}{2(y_2 - y_1)} - \frac{(x_3 - x_2)(x_3 + x_2)}{2(y_3 - y_2)};$$

whence, after substituting the values of x_1, y_1, etc., and making suitable reductions, we find

$$x = +\frac{c^2}{a}\cos\tfrac{1}{2}(p_2+p_3)\cos\tfrac{1}{2}(p_3+p_1)\cos\tfrac{1}{2}(p_1+p_2).$$

In like manner, by eliminating x,

$$y = -\frac{c^2}{b}\sin\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_3+p_1)\sin\tfrac{1}{2}(p_1+p_2).$$

Suppose (x_1, y_1) and (x_3, y_3) to approach (x_2, y_2), then the limiting position of the centre of the circumscribing will be the centre of curvature at (x_2, y_2). But at the limit,

$$p_1 = p_2 = p_3 = p, \quad \cos p = \frac{x}{a}, \quad \text{and} \quad \sin p = \frac{y}{b};$$

$$\therefore\ X = \frac{c^2x^3}{a^4}, \quad Y = -\frac{c^2y^3}{b^4}.$$

[*Charles Puryear.*]

99

FROM a full cask of wine a quantity is taken out at random and the cask filled with water, and then a quantity of the mixture is taken out at random and the cask again filled with water. What is the probability that the cask now contains more wine than water? [*Artemas Martin.*]

SOLUTION.

Let x = quantity of wine taken out at the first drawing, y = quantity of the mixture taken out at the second drawing. Then, a being the capacity of the cask, $a\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)$ = quantity of wine remaining in the cask after the second drawing. Putting

$$a\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right) = \tfrac{1}{2}a, \qquad (1)$$

we get

$$x = a\left\{1-\frac{1}{2\left(1-\frac{y}{a}\right)}\right\} = x_1.$$

Now the greater x is, the less the amount of wine left; hence x may have any value from 0 to x_1.

$$\therefore\ p' = \frac{x_1}{a} = 1 - \frac{1}{2\left(1-\frac{y}{a}\right)}$$

is the probability on the supposition that y is known.

The greatest value y can have is found by putting $x = 0$ in (1), which gives $y = \frac{1}{2}a$. Hence the required probability is

$$p = \frac{\int_0^{\frac{1}{2}a} p' dy}{\int_0^a dy} = \frac{1}{a}\int_0^{\frac{1}{2}a}\left\{1 - \frac{1}{2\left(1 - \frac{y}{a}\right)}\right\} dy$$

$$= \frac{1}{a}\left[y + \tfrac{1}{2}a \log\left(1 - \frac{y}{a}\right)\right]_0^{\frac{1}{2}a}$$

$$= 1 - \tfrac{1}{2}(1 + \log 2).$$

[*Artemas Martin.*]

101

EXPRESS in terms of $\sin^{-1}x$ and $\sin^{-1}y$

$$\tan^{-1}\frac{x+y}{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}.$$

SOLUTION.

If $\sin^{-1}x = a$ and $\sin^{-1}y = b$, we have

$$\tan^{-1}\frac{x+y}{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}} = \tan^{-1}\frac{\sin a + \sin b}{\cos a + \cos b}$$

$$= \tan^{-1}\tan\tfrac{1}{2}(a+b) = \tfrac{1}{2}(a+b) = \tfrac{1}{2}(\sin^{-1}x + \sin^{-1}y).$$

[*Cooper D. Schmitt.*]

EXERCISES.

111

A circle is inscribed in an isosceles triangle; another in the space between the first circle and the vertex, and so on *ad infinitum*. What is the vertical angle when the sum of the areas of the circles is $1/n$ of the area of the triangle?

[*L. G. Weld.*]

112

To inscribe in the given triangle ABC a triangle PQR whose sides QR, RP, PQ are perpendicular respectively to BC, CA, AB. [*Alfred C. Lane.*]

113

THE transversal MN meets the sides AB, AC of the fixed triangle ABC, and makes

$$AM \, . \, AN = BM \, . \, CN.$$

Find its envelope. [*O. Root, Jr.*]

114

A CIRCLE intersects a conic in four points, P_1, P_2, P_3, P_4. Show that if the x-axis be parallel to the axis of the conic the area of their quadrilateral is

$$(x_2 - x_4)(y_1 - y_3).$$ [*R. H. Graves.*]

115

IN an equilateral triangle inscribe a triangle similar to a given triangle and having a maximum area. [*W. M. Thornton.*]

116

FIND the value of $\int Qdx$ where

$$Q = \cos(a_1x + b_1)\cos(a_2x + b_2)\ldots\cos(a_nx + b_n),$$

$a_1, a_2, a_n, \ldots$ and $b_1, b_2, \ldots b_n$ being constants. [*A. Hall.*]

SELECTED.

117

SHOW that the angle between the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the tangent at the corresponding point of the principal circle is given by the formula,

$$\tan\delta = \frac{\alpha\tan\varphi}{(1-\alpha)+\tan^2\varphi},$$

where $\alpha = (a-b)/a$ is the ellipticity, and φ is the eccentric anomaly.

118

SHOW that the eccentric anomaly which gives the maximum deviation is $\tan^{-1}\sqrt{(1-\alpha)}$, and find the co-ordinates of the point of contact T and the equation to the tangent t.

119

SHOW that the portion of t intercepted between the axes equals in length the sum of the semi-axes a, b, and is divided at T into these two parts.

120

SHOW that the circumcircle of txy passes through the centre of curvature at T, and that the area of the circle of curvature equals that of the ellipse.

121

SHOW that when the ellipse varies subject to the condition, $a + b$ constant, the locus of T is the hypocycloid,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a+b)^{\frac{2}{3}},$$

which touches each ellipse at the point T.

CONTENTS.

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